Tens

For a sum to be a multiple of 10, the ones digits of the two terms must total to either 0 or 10 so that the ones digit of the sum is 0, thereby making it divisible by 10. Let us first consider the first case of 9^n+1^n. We can calculate the sum of the various sums of this expression for a few values of n.

Value of "n"	Value of "9^n"	Value of "1^n"	Sum of "9^n+1^n"
1	9	1	10
2	81	1	82
3	729	1	730
4	6561	1	6562
5	59049	1	59050
6	531441	1	521442

We can easily see here that since the ones digit of 9^n is 9 for odd values of n and 1 for even values of n and the ones digit of 1^n is always 1, the ones digits of the sum will be 0 for all odd values of n and 2 for all even values of n. This can be proven by the fact that the ones digit of 9^n is solely determined by the ones digit of 9^{n-1}. Therefore, since 9x9 gives us a ones digit of 1 and 1x9 gives us a ones digit of 9, we have a recurring sequence of ones digits oscillating between 9 and 1.

Therefore, 9^n+1^n is divisible by 10 for all odd n.

Now that we recognize the significance of the ones digit in determining whether the sum of expressions in the form a^n+b^n are divisible by ten, it follows to construct a table to view the patterns of the ones digits when the 10 digits are raised to a power n.

n	1^n	2^n	3^n	4^n	5^n	6^n	7^n	8^n	9^n	0^n
1	1	2	3	4	5	6	7	8	9	0
2	1	4	9	6	5	6	9	4	1	0
3	1	8	7	4	5	6	3	2	9	0
4	1	6	1	6	5	6	1	6	1	0
5	1	2	3	4	5	6	7	8	9	0
6	1	4	9	6	5	6	9	4	1	0
7	1	8	7	4	5	6	3	2	9	0
8	1	6	1	6	5	6	1	6	1	0

We can then determine the period of each series a^n (period is the number of multiplications by itself that a number "a" needs to undergo such that the product has the same ones digit as "a"). We can illustrate these in a table (as shown below):

	1	2	3	4	5	6	7	8	9	0
Period	1	4	4	2	1	1	4	4	2	1

We can then conclude that numbers with the same period (e.g. 8 and 3) when raised to a power n and then added together will produce a sum with a constant pattern of ones digits. So by this and looking at the figures from the table, we can conclude that:

- **9^n+1^n** is divisible by 10 for all odd n (with a last digit of 1+9=0) and divisible by 2 for all even n (with a last digit 1+1=2)
- **7^n+3^n** is divisible by 10 for all odd n (with a last digit of either 3+7 or 7+3=0) and divisible by 2 for all even n (with a last digit of 9+9=8 for all n=2r (where "r" is odd) and a last digit of 1+1=2 for all n=2r (where r is even))
- **6^n-4^n** is divisible by 10 for all even n (with a last digit of 6-6=0) and divisible by 2 for all odd n (with a last digit of 6-4=0)

- **8^n-2^n** is divisible by 10 for all even n (with a last digit of either 4-4 or 6-6=0) and divisible by 2 for all odd n (with a last digit of 8-2=6 for all n=2r-1 (where “r” is odd) and a last digit of 2-8=4 for all n=2r-1 (where r is even))

PROOF USING BINOMIAL THEOREM

We know the binomial theorem, which states that:

$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$

and since we know a+b=10, $(a+b)^n$ must be divisible by 10

We also know that the expansion of $(a+b)^n$ will give us n+1 terms, so if we let n be odd, we get an even number of terms in the expansion which allows us to group the terms as follows with the knowledge that $\binom{x}{a} = \binom{x}{x-a}$:

$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}ab^{n-1} + b^n$

$= (a^n+b^n) + \binom{n}{1}ab(a^{n-2}+b^{n-2}) + \binom{n}{2}a^2b^2(a^{n-4}+b^{n-4}) + \ldots + \binom{n}{(n-1)/2}a^{(n-1)/2}b^{(n-1)/2}(a+b)$

Therefore, since (a+b) is divisible by 10, we can prove that (a^3+b^3) is divisible by 10 by substituting n=3 to get:

$(a+b)^3 = (a^3+b^3) + \binom{n}{1}ab(a+b)$ and since (a+b) is divisible by 10, $(a+b)^3$ is divisible by 10 and thus, a^3+b^3 is divisible by 10 and through a similar method, we can prove that subsequent sums of a^n+b^n are divisible by 10 for all odd n.

Therefore, for all odd n, a^n+b^n is divisible by 10 if a+b=10

Now if we consider a^n-b^n, for an even n, we can express this as

$a^n-b^n = (a^{n/2}+b^{n/2})(a^{n/2}-b^{n/2})$ and since we know that $(a^{n/2}+b^{n/2})$ is divisible by 10 as $\frac{n}{2}$ is odd, then a^n-b^n must also be divisible by 10

Therefore, for all even n, a^n-b^n is divisible by 10 if a+b=10